Baire Δ_1-spaces and Asplund spaces $C_k(X)$ over Δ_1-spaces X

JERZY KĄKOL

A. MICKIEWICZ UNIVERSITY, POZNAŃ

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Theorem 4 (Jayne–Rogers–Ribarska, Namioka)

A Banach space E is Asplund iff $(B_{E'}, w^*)$ is fragmented by the metric generated by the dual norm.

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3. Similar locally convex versions of other Banach spaces properties (like (NP)-property) were introduced and studied by Komisarchik and Megrelishvili (2023).

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7. If X is k-scattered, then $C_k(X)$ is a (df)-space (in sense of Jarchow) iff X is ω-bounded (Mazon).

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Recall the following two concepts; some discussion below.

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Recall the following two concepts; some discussion below.

Definition 7 (Kąkol–Leiderman)

A topological space X is a Δ-space (Δ_1-space) if for every decreasing sequence $(D_n)_n$ of (countable) subsets of X with $\bigcap_n D_n = \emptyset$, there is a decreasing sequence $(V_n)_n$ of open subsets of X, $D_n \subset V_n$ for every $n \in \omega$ and $\bigcap_n V_n = \emptyset$.

Theorem 8 (Kąkol–Kurek–Leiderman)

A pseudocompact X is a Δ_1-space iff every countable set is scattered. If X is a Cech-complete space, then X is scattered iff X is in Δ_1.

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Corollary 9

Let X be a compact space. The following assertions are equivalent: (i) The space X is a Δ_1-space. (ii) The space X is scattered. (iii) The space $C_k(X)$ is an Asplund space.

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Problem 10

Let X be a compact space. Find a "nice" property $\mathcal{P}$ on $C(X)$ or $C(X)'$ under which the following statement holds true. X is a Δ-space iff $C(X)$ is Asplund and $C(X)$ (or $C(X)'$) satisfies property $\mathcal{P}$.

1. The class of Čech-complete spaces is "disjoint" from the class of ω-bounded spaces, so both cases are mutually complementary describing the Asplund property for $C_k(X)$.

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2. In Theorem 6 the assumption on X cannot be removed. Indeed, Juhasz and van Mill constructed a countably compact (not ω-bounded and) not scattered space $X \subset \beta\omega \setminus \omega$ with all countable subsets scattered, so every compact subset of X is scattered.

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3. Nevertheless, the discussed space X is not Warner bounded but it is a Δ_1-space.

Theorem 6 motivates the following sharper example due to Marciszewski.

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Example 11

There exists a Warner bounded set X such that X is not ω-bounded and every compact subset of X is scattered but X is not scattered.

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8. No Δ-set X has cardinality $\mathfrak{c}$ (Przymusiński (1977)).

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5. A compact Eberlein space is a Δ-space iff it is scattered.
6. Every pseudocompact Δ_1-space with countable tightness is scattered (J.K.-Kurka-Leiderman).

On ω-resolvable Δ-spaces and Δ_1-spaces

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3. One of the problems considered by Malykhin refers to the existence of irresolvable spaces (i.e. crowded not resolvable) satisfying the Baire Category Theorem. Under L=V (i.e. every set is constructible) there is no Baire irresolvable space (Kunen-Szymański-Tall).

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4. (V=L) $\Rightarrow$ (AC) $\wedge$ (GCH).

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If X is Baire and ω-resolvable, then X is not a Δ-space. Hence, a crowded Lindelöf Baire space is not a Δ-space.

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3. Consider some cases for which this problem has a positive answer.

1. Under Souslin hypothesis (SH) (i.e. there are no Souslin lines) if X is crowded and Baire with the cellularity $c(X) \leq \aleph_0$, then X is ω-resolvable (Casarrubias-Segura, Hernandez-Hernandez, Tamaris-Mascar) (Recall that $(MA) \wedge (\sim (CH)) \Rightarrow (SH)$.)

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(SH) If X is a Baire space with $c(X) \leq \aleph_0$ and X is a Δ-space, then X has isolated points.

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Corollary 14

(SH) If X is a Baire space with $c(X) \leq \aleph_0$ and X is a Δ-space, then X has isolated points.

2. The axiom of constructibility, $V = L$, implies that every Baire space without isolated points is ω-resolvable (Pavlov). Hence (under $V = L$) every Δ-space which is Baire has isolated points.

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4. X is a Baire space and $c(X) \leq \aleph_0$.

Corollary 15

(MA) If X is a Baire space which is a Δ-space and that satisfies one of the mentioned above properties $\mathcal{P}$, then X has isolated points.

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Example 17

There exist crowded countably compact Δ_1-spaces not Δ-spaces which are ω-resolvable.

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Problem 18 (J.K.-Leiderman-Tkachuk)

Does a metrizable Baire Δ_1-space admit isolated points?

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Does a metrizable Baire Δ_1-space admit isolated points?

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3. Do there exist in (ZFC) non-normal separable first countable countably compact Hausdorff spaces (Nyikos)?